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New Maximum Principles for First-Order Impulsive Boundary Value Problems

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Abstract—In this paper, we establish new maximum principles for a boundary value problem for first-order impulsive differential equations.

Keywords—Maximum principle, Impulsive equation, Boundary value problem.

1. INTRODUCTION AND PRELIMINARIES

Maximum principles for boundary value problems of ordinary differential equations without impulsive effects have been extensively studied in the literature. It is well known that maximum principles play an important role in the theory of differential equations. They often are employed to study some qualitative aspects of differential equations. They are also essential for developing the monotone iterative method, a powerful theoretical method [1], which permits us to construct a sequence of approximate solutions converging to a solution of certain differential equations problems.

On the other hand, "impulsive effects" should be and have been incorporated into realistic models in many applications. Indeed, differential equations with impulses are a basic tool for studying evolution processes that are subject to abrupt changes in their states (we refer to [2]). Therefore, it is of the utmost importance to develop a general theory for differential equations with impulses including some basic aspects of this theory (cf., [2]).

In the present paper, we consider the following first-order impulsive boundary value problem:

$$x'(t) + \lambda x(t) = q(t), \qquad t \neq t_k, \quad t \in J = [0,T], \quad k = 1,\dots,m, \tag{1}$$

$$x(0) = x(T) + \mu, \tag{2}$$

$$x\left(t_k^+\right) = L_k\left(x\left(t_k\right)\right), \qquad k = 1,\dots,m, \tag{3}$$

where $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\lambda, \mu \in R$, $L_k \in C(R,R)$, $k = 1,2,\dots,m$, are some nonlinear functions, and $q : J \to R$ is such that $q|_{(t_k,t_{k+1})}$ is continuous, there exist the limits $q(t_k^-) = \lim_{h\to 0+} q(t_k - h)$, $q(t_k^+) = \lim_{h\to 0+} q(t_k + h)$, and $q(t_k^-) = q(t_k)$ for each $k = 1,\dots,m$.

We will refer to problems (1)–(3) as an IBVP and present new maximum principles which generalize/improve previous known results. We first introduce the following spaces of functions

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so that we can define more precisely the concept of solutions for an IBVP. Set $PC(J) = \{x : J \to R;\ x|_{(t_k,t_{k+1})} \in C(t_k,t_{k+1}),\ k = 0,1,\dots,m,\ \exists\, x(0^+),\ x(T^-),\ x(t_k^+), x(t_k^-),\ x(t_k^-) = x(t_k),\ k = 1,2,\dots,m\}$, and $PC^1(J) = \{x \in PC(J);\ x|_{(t_k,t_{k+1})} \in C^1(t_k,t_{k+1}),\ k = 0,1,\dots,m,\ \exists\, x'(0^+),\ x'(T^-),\ x'(t_k^+),\ x'(t_k^-),\ k = 1,2,\dots,m\}$.

It is clear that $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms

$$\|x\|_{PC(J)} = \sup\{|x(t)|;\ t \in J\} \qquad \text{and} \qquad \|x\|_{PC^1(J)} = \|x\|_{PC(J)} + \|x'\|_{PC(J)}.$$

By a solution of an IBVP, we mean a function $x \in PC^1(J)$ which satisfies equation (1) for every $t \in J - \{t_1,\dots,t_m\}$ and the boundary condition (2), and at every t_k, $k = 1,\dots,m$, the function x satisfies (3).

Recall that in the nonimpulsive case, that is, if $L_k(x) = x$, $k = 1,\dots,m$, then $x(t_k^+) = x(t_k)$ and $x \in C[0,T]$ (we say BVP). It is well known that a BVP, in this case, has a unique solution for any $q \in C(J)$ and $\mu \in R$ if and only if $\lambda \neq 0$. While in the case $\lambda = 0$, the problem is solvable if and only if $\int_0^T q(t)\,dt = 0$. In such a case, there exists an infinite number of solutions. In other words, the only eigenvalue of x' with periodic conditions is $\lambda = 0$ (the eigenvalues are constants). For the nonimpulsive case, the following maximum principle which depends on the sign of $\lambda \neq 0$ is well known.

THEOREM A. *Assume in the BVP that $q \in C(J)$. Then we have the following.*

(a) *If $q(t) \geq 0$ in J and $\mu \geq 0$, then*

$$\begin{aligned} \lambda > 0 &\Longrightarrow x \geq 0, \\ \lambda < 0 &\Longrightarrow x \leq 0. \end{aligned}$$

(b) *If $q(t) \leq 0$ in J and $\mu \leq 0$, then*

$$\begin{aligned} \lambda > 0 &\Longrightarrow x \leq 0, \\ \lambda < 0 &\Longrightarrow x \geq 0. \end{aligned}$$

In the impulsive case, it is to be pointed out that an IBVP is not always solvable (even if $\lambda \neq 0$, and L_k are linear functions). See the examples in [3].

However, we can give sufficient and necessary conditions for an IBVP to have a unique solution. As a simple decript, we consider an IBVP with $m = 1$ and establish the following proposition.

PROPOSITION 1. *Consider an IBVP with $m = 1$, then it has a unique solution for any $q \in PC(J)$ if and only if the root of the algebraic equation*

$$z = e^{-\lambda T} L_1(z + \alpha) + \beta \tag{4}$$

is real and unique, where

$$\begin{aligned} \alpha &= \int_0^{t_1} e^{-\lambda(t_1 - s)} q(s)\,ds, \\ \beta &= e^{-\lambda t_1} \left(\int_{t_1}^{T} e^{-\lambda(T-s)} q(s)\,ds + \mu \right). \end{aligned}$$

PROOF. Let $x(0) = x_0$ be given. With this initial condition and equations (1) and (3), one has a Cauchy problem that is solvable and it has a unique solution x for each x_0. For $0 \leq t \leq t_1$, we have

$$\begin{aligned} x(t) &= e^{-\lambda t} x(0) + \int_0^t e^{-\lambda(t-s)} q(s)\,ds, \\ x(t_1^+) &= L_1 \left(e^{-\lambda t_1} x(0) + \int_0^{t_1} e^{-\lambda(t_1-s)} q(s)\,ds \right). \end{aligned}$$

For $t_1 < t \leq T$, we have

$$\begin{aligned} x(t) &= e^{-\lambda(t-t_1)}x\left(t_1^+\right) + \int_{t_1}^{t} e^{-\lambda(t-s)}q(s)\,ds \\ &= e^{-\lambda(t-t_1)}L_1\left(e^{-\lambda t_1}x(0) + \int_0^{t_1} e^{-\lambda(t_1-s)}q(s)\,ds\right) + \int_{t_1}^{t} e^{-\lambda(t-s)}q(s)\,ds. \end{aligned}$$

Now, a solution x of the Cauchy problem will be a solution of an IBVP if and only if it satisfies the boundary condition $x(0) = x(T) + \mu$. Then, from the last equality, we have

$$x(0) - \mu = e^{-\lambda(T-t_1)}L_1\left(e^{-\lambda t_1}x(0) + \int_0^{t_1} e^{-\lambda(t_1-s)}q(s)\,ds\right) + \int_{t_1}^{T} e^{-\lambda(T-s)}q(s)\,ds.$$

Thus, the condition that (4) has a unique real root implies the unique existence of an initial value $x(0)$ satisfying the boundary condition for any $q \in PC(J)$. The proof is complete.

Note that an IBVP is not really a linear problem, since the impulse functions L_k are not necessarily linear. However, if L_k $(k = 1, \ldots, m)$ are linear, then an IBVP is a linear impulse problem. In particular, in the case when $L_k(x) = c_k x$, the next result gives an explicit formula for the unique solution of an IBVP.

PROPOSITION 2. *Consider an IBVP with $L_k(x) = c_k x$, $k = 1, \ldots, m$. Then an IBVP admits a unique solution for all $q \in PC(J)$ if and only if $\prod_{k=1}^m c_k \neq e^{\lambda T}$. Moreover, the unique solution is given by*

$$\begin{aligned} x(t) = e^{-\lambda t}\left(\prod_{0<t_k<t} c_k\right)\left[\mu + \int_0^T \left(\prod_{s<t_k<T} c_k\right) e^{-\lambda(T-s)}q(s)\,ds\right]\left(1 - e^{-\lambda T}\prod_{k=1}^m c_k\right)^{-1} \\ + \int_0^t \left(\prod_{s<t_k<t} c_k\right) e^{-\lambda(t-s)}q(s)\,ds. \end{aligned} \tag{5}$$

PROOF. Let $x(0) = x_0$ be given. Then with this initial condition and equations (1) and (3), we have a Cauchy problem which is solvable and it has a unique solution x for each x_0. By Theorem 1.5.1 in [2], we have, for $0 \leq t \leq T$,

$$x(t) = x_0\left(\prod_{0<t_k<t} c_k\right) e^{-\lambda t} + \int_0^t \left(\prod_{s<t_k<t} c_k\right) e^{-\lambda(t-s)}q(s)\,ds. \tag{6}$$

In particular, for $t = T$, we have

$$x(T) = x_0 e^{-\lambda T}\prod_{k=1}^m c_k + \int_0^T \left(\prod_{s<t_k<T} c_k\right) e^{-\lambda(T-s)}q(s)\,ds. \tag{7}$$

Now, a solution of the Cauchy problem will be a solution of an IBVP if and only if it satisfies the boundary condition $x(0) = x(T) + \mu$. Thus, by (7), we have

$$\left(1 - e^{-\lambda T}\prod_{k=1}^m c_k\right)x(0) = \mu + \int_0^T \left(\prod_{s<t_k<T} c_k\right) e^{-\lambda(T-s)}q(s)\,ds. \tag{8}$$

Thus, for every $q \in PC(J)$, there exists an initial condition $x(0)$ satisfying the boundary condition if and only if $\prod_{k=1}^m c_k \neq e^{\lambda T}$. Moreover, in this case, from (6) and (8), we have that, for $0 \leq t \leq T$, (5) holds. The proof is complete.

In [2], the following maximum principle for an IBVP with $L_k(x) = c_k x$ was established which partially generalizes Theorem A to the responding impulsive problem.

THEOREM B. *Consider the problem of an IBVP with $L_k(x) = c_k x$. Assume that $\lambda > 0$, $\mu \geq 0$, $q(t) \geq 0$ in J, $c_k > 0$, $k = 1, \dots, m$, and $\prod_{k=1}^{m} c_k < e^{\lambda T}$. Then $x(t) \geq 0$ for $t \in J$.*

It should be noted that, in Theorem B, there is no information about the sign of $x(t)$ when either $\lambda < 0$ or $\prod_{k=1}^{m} c_k > e^{\lambda T}$.

We now define the operator $\Gamma : D(\Gamma) \to PC(J)$ by

$$\Gamma(x)(t) = x'(t), \quad t \neq t_k, \qquad \Gamma(x)(t_k) = x'(t_k^-), \quad k = 1, \dots, m,$$

where

$$D(\Gamma) = \left\{ x \in PC^1(J) : x(t_k^+) = c_k x(t_k),\ k = 1, \dots, m,\ x(0) = x(T) + \mu \right\}.$$

We observe that the problem IBVP with $L_k(x) = c_k x$, $\mu = 0$, and $\lambda = 0$ is equivalent to the abstract equation

$$\Gamma x = q, \qquad x \in D(\Gamma).$$

Theorem B can be seen as a sufficient condition to assure that the operator $(\Gamma + \lambda I)$ is inverse positive, that is, $(\Gamma + \lambda I)(x) \geq 0$ on J implies that $x \geq 0$ on J.

2. MAIN RESULTS

In this section, we will first present a new maximum principle for an IBVP under the following assumptions:

$$L_k(x) \geq 0, \quad \text{for } x \geq 0, \qquad L_k(x) \leq c_k x, \quad \text{for } x < 0, \tag{9}$$

$$L_k(x) X \leq 0, \quad \text{for } x \leq 0, \qquad L_k(x) \geq c_k x, \quad \text{for } x > 0, \tag{10}$$

where $k = 1, 2, \dots, m$. We note that this result does not need the sign of $\prod_{k=1}^{m} c_k - e^{\lambda T}$.

THEOREM 2.1. *Consider the problem IBVP. Let $x(t)$ be a solution. Then we have the following.*

(a) *If (9) is satisfied, $\lambda > 0$, $\mu \geq 0$, $q(t) \geq 0$, $t \in J$, and $c_k > 1$, $k = 1, \dots, m$, then $x(t) \geq 0$ in J.*

(b) *If (10) is satisfied, $\lambda > 0$, $\mu \leq 0$, $q(t) \leq 0$, $t \in J$, and $c_k > 1$, $k = 1, \dots, m$, then $x(t) \leq 0$ in J.*

PROOF OF THEOREM 2.1(a). Let $s \in J$ such that

$$x(s) = \min_{t \in J} x(t).$$

Suppose $x(s) < 0$. We first claim that $s \neq t_k$, $k = 1, \dots, m$. Otherwise, if $s = t_i$ for some $i \in \{1, \dots m\}$, then

$$x\left(t_i^+\right) = L_i\left(x\left(t_i\right)\right) \leq c_i x\left(t_i\right) < x\left(t_i\right).$$

This implies that there exists $s' \in [0, T]$ such that $x(s') < x(t_i)$, a contradiction. Next, we assume $s = T$. In such a case, if $x(t_m) \geq 0$, then $x(t_m^+) = L_m(x(t_m)) \geq 0$. Thus, there exists $s_1 \in [t_m, T)$ such that $x(s_1^+) = 0$, $x(t) < 0$, $t \in (s_1, T]$. The mean value theorem implies that there exists $s_2 \in (s_1, T)$ such that $x'(s_2) < 0$ and $x(s_2) < 0$. But in this situation, we are led to the contradiction

$$0 \leq q(s_2) = x'(s_2) + \lambda x(s_2) < 0.$$

If $x(t_m) < 0$, then $x(t_m^+) \leq c_m x(t_m) < 0$. Thus, there exists $s_0 \in (t_m, T)$ such that $x'(s_0) \leq 0$, $x(s_0) < 0$. But in this situation, we again obtain the contradiction

$$0 \leq q(s_0) = x'(s_0) + \lambda x(s_0) < 0.$$

Now, if $s = 0$, then $s = T$, since $\mu \geq 0$. Finally, if $s \in \int(J)$, and $s \neq t_k$, $k = 1, \dots, m$, then $x'(s) = 0$ and so

$$0 \leq q(s) = x'(s) + \lambda x(s) < 0.$$

This is impossible. Thus, $x(t) \geq 0$ in J.

PROOF OF THEOREM 2.1(b). Let $s \in J$ such that

$$x(s) = \max_{t \in J} x(t).$$

Suppose $x(s) > 0$. We first claim that $s \neq t_k$, $k = 1, \dots, m$. Otherwise, if $s = t_i$ for some $i \in \{1, \dots, m\}$, then

$$x\left(t_i^+\right) = L_i\left(x\left(t_i\right)\right) \geq c_i x\left(t_i\right) > x\left(t_i\right).$$

Thus, there exists $s' \in [0, T]$ such that $x(s') > x(t_i)$, a contradiction. Next, we assume $s = T$. If $x(t_m) \leq 0$, then $x(t_m^+) = L_m(x(t_m)) \leq 0$. Thus, there exists $s_1 \in [t_m, T)$ such that $x(s_1^+) = 0$, $x(t) > 0$, $t \in (s_1, T)$. The mean value theorem implies that there exists $s_2 \in (s_1, T)$ such that $x'(s_2) > 0$ and $x(s_2) > 0$. But in this situation, we obtain the contradiction

$$0 \geq q(s_2) = x'(s_2) + \lambda x(s_2) > 0.$$

If $x(t_m) > 0$, then $x(t_m^+) \geq c_m x(t_m) > 0$. Thus, there exists $s_0 \in (t_m, T)$ such that $x'(s_0) \geq 0$, $x(s_0) > 0$. But in this situation, we again obtain the contradiction

$$0 \geq q(s_0) = x'(s_0) + \lambda x(s_0) > 0.$$

Now, if $s = 0$, then $s = T$ since $\mu \leq 0$. Finally, if $s \in \int(J)$ and $s \neq t_k$, $k = 1, \dots, m$, then $x'(s) = 0$, and so

$$0 \geq q(s) = x'(s) + \lambda x(s) > 0.$$

This is impossible. Therefore, $x(t) \leq 0$ in J. The proof is complete.

In the following, we explain how to use impulsive differential inequalities to obtain a maximum principle for an IBVP which only depends on the sign of $\prod_{k=1}^{m} c_k - e^{\lambda T}$. Note that, in this case, the sign of λ is not important. We need the following lemma which is a particular case of Corollary 1.4.1 in [2].

LEMMA 2.1. *Let $s \in [0, T)$, $c_k \geq 0$, α_k, $k = 1, \dots, m$ be constants and let $p, q \in PC(J)$, $x \in PC^1(J)$.*

(a) *If*

$$\begin{aligned} x'(t) &\leq p(t)x(t) + q(t), && t \in [s, T), \quad t \neq t_k, \\ x(t_k^+) &\leq c_k x(t_k) + \alpha_k, && t_k \in [s, T), \end{aligned}$$

then, for $t \in [s, T)$,

$$\begin{aligned} x(t) \leq\ & x\left(s^+\right) \left(\prod_{s < t_k < t} c_k \right) \exp \int_s^t p(u)\, du \\ &+ \int_s^t \left(\prod_{u < t_k < t} c_k \right) \exp \left(\int_u^t p(\tau)\, d\tau \right) q(u)\, du \\ &+ \sum_{s < t_k < t} \left(\prod_{t_k < t_i < t} c_i \right) \exp \left(\int_{t_k}^t p(\tau)\, d\tau \right) \alpha_k. \end{aligned}$$

(b) *If*

$$\begin{aligned} x'(t) &\geq p(t)x(t) + q(t), && t \in [s,T), \quad t \neq t_k, \\ x\left(t_k^+\right) &\geq c_k x\left(t_k\right) + \alpha_k, && t_k \in [s,T), \end{aligned}$$

then for $t \in [s,T)$,

$$\begin{aligned} x(t) \geq\ & x\left(s^+\right)\left(\prod_{s<t_k<t} c_k\right)\exp\int_s^t p(u)\,du \\ &+ \int_s^t \left(\prod_{u<t_k<t} c_k\right)\exp\left(\int_u^t p(\tau)\,d\tau\right) q(u)\,du \\ &+ \sum_{s<t_k<t}\left(\prod_{t_k<t_i<t} c_i\right)\exp\left(\int_{t_k}^t p(\tau)\,d\tau\right)\alpha_k. \end{aligned}$$

THEOREM 2.2. *Consider the problem IBVP. Assume that there exist constants* $c_k > 0$ *and* α_k *such that*

$$L_k(x) > 0a(x > 0), \quad L_k(x) \geq c_k x + \alpha_k, \quad (x \in R), \qquad k = 1, \dots, m. \tag{11}$$

If $\mu \geq 0$ *and, for* $0 \leq s' \leq t \leq T$,

$$\int_{s'}^t \left(\prod_{s<t_k<t} c_k\right) e^{\lambda s} q(s)\,ds + \sum_{s'<t_k<t}\left(\prod_{t_k<t_i<t} c_i\right) e^{\lambda t_k}\alpha_k \geq 0, \tag{12}$$

then

$$\begin{aligned} \prod_{k=1}^m c_k < e^{\lambda T} &\Longrightarrow x(t) \geq 0, \\ \prod_{k=1}^m c_k > e^{\lambda T} &\Longrightarrow x(t) \leq 0. \end{aligned}$$

PROOF. From Lemma 2.1 and (12), if $x \in PC^1(J)$ is a solution of an IBVP, then for $t \in J$, x satisfies

$$\begin{aligned} x(t) &\geq x(0)\left(\prod_{0<t_k<t} c_k\right) e^{-\lambda t} + \int_0^t \left(\prod_{s<t_k<t} c_k\right) e^{-\lambda(t-s)} q(s)\,ds \\ &\quad + \sum_{0<t_k<t}\left(\prod_{t_k<t_i<t} c_i\right) e^{-\lambda(t-t_k)}\alpha_k \\ &\geq x(0)\left(\prod_{0<t_k<t} c_k\right) e^{-\lambda t}. \end{aligned} \tag{13}$$

If $\prod_{k=1}^m c_k < e^{\lambda T}$, by (13), it is sufficient to prove that $x(0) \geq 0$. Suppose $x(0) < 0$, then by (13), we have

$$x(0) = x(T) + \mu \geq x(T) \geq x(0)\left(\prod_{k=1}^m c_k\right) e^{-\lambda T} > x(0),$$

a contradiction. Thus, $x(t) \geq 0$ on J.

If $\prod_{k=1}^{m} c_k > e^{\lambda T}$, we first claim that $x(0) \le 0$. Suppose $x(0) > 0$, then by (13), we have

$$x(0) = x(T) + \mu \ge x(T) \ge x(0) \left(\prod_{k=1}^{m} c_k \right) e^{-\lambda T} > x(0),$$

a contradiction, and so $x(0) \le 0$. To prove that $x(t) \le 0$ on J, assume that there exists $s \in J$ such that $x(s) > 0$. Then $s < T$. If $s = t_i$ for some $i \in \{1, \dots, m\}$, then $x(s^+) = L_i(x(s)) > 0$. Thus, in both the cases $s = t_i$ for some $i \in \{1, \dots, m\}$ and $s \neq t_k$, $k = 1, \dots, m$, we have $x(s^+) > 0$. Now, by Lemma 2.1 and (12), it follows that

$$x(t) \ge x\left(s^+\right) \left(\prod_{s<t_k<t} c_k \right) e^{-\lambda(t-s)}, \qquad s < t \le T.$$

In particular, for $t = T$, we have

$$x(T) \ge x\left(s^+\right) \left(\prod_{s<t_k<T} c_k \right) e^{-\lambda(T-s)} > 0.$$

Thus, $x(0) = x(T) + \mu \ge x(T) > 0$. This is a contradiction and so $x(t) \le 0$. The proof is complete.

THEOREM 2.3. *Consider the problem IBVP. Assume that there exist constants $c_k > 0$ and α_k such that*

$$L_k(x) < 0 \ (x < 0), \quad L_k(x) \le c_k x + \alpha_k \ (x \in R), \qquad k = 1, \dots, m.$$

If $\mu \le 0$ and, for $0 \le s' \le t \le T$,

$$\int_{s'}^{t} \left(\prod_{s<t_k<t} c_k \right) e^{\lambda s} q(s)\, ds + \sum_{s'<t_k<t} \left(\prod_{t_k<t_i<t} c_i \right) e^{\lambda t_k} \alpha_k \le 0, \tag{14}$$

then

$$\prod_{k=1}^{m} c_k < e^{\lambda T} \Longrightarrow x(t) \le 0,$$
$$\prod_{k=1}^{m} c_k > e^{\lambda T} \Longrightarrow x(t) \ge 0.$$

PROOF. From Lemma 2.1 and (14), if $x \in PC^1(J)$ is a solution of an IBVP, then for $t \in J$, x satisfies

$$\begin{aligned} x(t) &\le x(0) \left(\prod_{0<t_k<t} c_k \right) e^{-\lambda t} + \int_0^t \left(\prod_{s<t_k<t} c_k \right) e^{-\lambda(t-s)} q(s)\, ds \\ &\quad + \sum_{0<t_k<t} \left(\prod_{t_k<t_i<t} c_i \right) e^{-\lambda(t-t_k)} \alpha_k \\ &\le x(0) \left(\prod_{0<t_k<t} c_k \right) e^{-\lambda t}. \end{aligned} \tag{15}$$

If $\prod_{k=1}^{m} c_k < e^{\lambda T}$, by (15), it is sufficient to prove that $x(0) \le 0$. Suppose $x(0) > 0$, then by (15), we have

$$x(0) = x(T) + \mu \le x(T) \le x(0) \left(\prod_{k=1}^{m} c_k \right) e^{-\lambda T} < x(0),$$

a contradiction. Thus, $x(t) \le 0$ on J.

If $\prod_{k=1}^{m} c_k > e^{\lambda T}$, we first claim that $x(0) \geq 0$. Suppose $x(0) < 0$, then by (15), we have

$$x(0) = x(T) + \mu \leq x(T) \leq x(0) \left(\prod_{k=1}^{m} c_k \right) e^{-\lambda T} < x(0),$$

a contradiction and so $x(0) \geq 0$. To prove that $x(t) \geq 0$ on J, assume that there exists $s \in J$ such that $x(s) < 0$. Then $s < T$. If $s = t_i$ for some $i \in \{1, \ldots, m\}$, then $x(s^+) = L_i(x(s)) < 0$. Thus, in both the cases $s = t_i$ for some $i \in \{1, \ldots, m\}$ and $s \neq t_k$, $k = 1, \ldots, m$, we have $x(s^+) < 0$. Now, by Lemma 2.1 and (14), it follows that

$$x(t) \leq x\left(s^+\right) \left(\prod_{s<t_k<t} c_k \right) e^{-\lambda(t-s)}, \qquad s < t \leq T.$$

In particular, for $t = T$, we have

$$x(T) \leq x\left(s^+\right) \left(\prod_{s<t_k<T} c_k \right) e^{-\lambda(T-s)} < 0.$$

Thus, $x(0) = x(T) + \mu \leq x(T) < 0$. This is a contradiction and so $x(t) \geq 0$. The proof is complete.

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